TUTORIAL - 5

1. If x and y two rvs having joint density function

$$f(x,y) = \begin{cases} \frac{1}{8}(6-x-y), & 0<x<2,\ 2<y<4 \\ 0, & \text{otherwise} \end{cases}$$

i) Find $P(X<1 \cap Y<3)$

$$P(X<1, Y<3) = \frac{1}{8}\int_2^3\int_0^1 (6-x-y)\,dx\,dy = \frac{1}{8}\int_2^3 \left(6x-\frac{x^2}{2}-xy\right)_0^1 dy$$

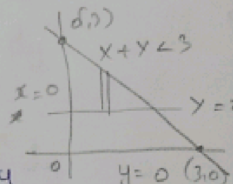
$$= \frac{1}{8}\int_2^3 \left(6-\frac{1}{2}-y\right)dy$$

$$= \frac{1}{8}\left(6y-\frac{y}{2}-\frac{y^2}{2}\right)_2^3$$

$$= \frac{1}{8}\left[18-\frac{3}{2}-\frac{9}{2}-(12-1-2)\right]$$

$$= \frac{1}{8}(18-9-6)$$

$$= \frac{3}{8}$$



ii) $$P(X+Y<3) = \int_2^3\int_0^{3-y} f(x,y)\,dx\,dy$$

$$= \int_2^3\int_0^{3-y} \frac{1}{8}(6-x-y)\,dx\,dy = \frac{1}{8}\int_2^3 \left(6x-\frac{x^2}{2}-xy\right)_0^{3-y} dy$$

$$= \frac{1}{8}\int_2^3 \left[6(3-y)-\frac{(3-y)^2}{2}-(3-y)y\right]dy$$

$$= \frac{1}{8}\int_2^3 \left(18-6y-\frac{1}{2}(9+y^2-6y)-3y+y^2\right)dy$$

$$= \frac{1}{8}\left[18y-\frac{6y^2}{2}-\frac{9y}{2}-\frac{y^3}{6}+\frac{6y^2}{4}-\frac{3y^2}{2}+\frac{y^3}{3}\right]_2^3$$

$$= \frac{1}{8}\left[54-27-\frac{27}{2}-\frac{27}{6}+\frac{27}{2}+\frac{27}{2}+\frac{27}{3}-\left(36-12-9-\frac{8}{6}+\frac{12}{2}-\frac{12}{2}+\frac{8}{3}\right)\right]$$

$$= \frac{5}{24}$$

iii) $$P(X<1 / Y<3) = \frac{P(X<1, Y<3)}{P(Y<3)}$$

$$\cancel{P(Y<3) = \int_2^3 f(y)\,dy} = \cancel{\frac{1}{8}\int}$$

$$P_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \frac{1}{8}\int_0^{\infty}(6-x-y)\,dx$$

$$= \frac{1}{8}\left(6x - \frac{x^2}{2} - xy\right)_0^2$$

$$= \frac{1}{8}(12 - 2 - 2y - (0))$$

$$= \frac{10-2y}{8}$$

$$P(Y<3) = \int_2^3 f_Y(y)\,dy = \frac{1}{8}\int_2^3 (10-2y)\,dy$$

$$= \frac{1}{8}\left(10y - 2\frac{y^2}{2}\right)_2^3$$

$$= \frac{1}{8}(30 - 9 - 20 + 4)$$

$$P(Y<3) = \frac{5}{8}$$

$$P(X<1 / Y<3) = \frac{3/8}{5/8} = \frac{3}{5}.$$

2. If the joint pdf of (X,Y) is $f(x,y) = \begin{cases} \frac{6}{5}(x+y^2), & 0 \le x \le 1, \; 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$

Find $P\left(\frac{1}{4} \le y \le \frac{3}{4}\right)$.

Marginal density of $Y = f(y) = \int_0^1 f(x,y)\,dx$

$$= \int_0^1 \frac{6}{5}(x+y^2)\,dx$$

$$= \frac{6}{5}\left(\frac{x^2}{2} + xy^2\right)_0^1$$

$$= \frac{6}{5}\left(\frac{1}{2} + y^2 - 0\right)$$

$$f(y) = \frac{6}{5}\left(\frac{1}{2} + y^2\right), \; 0 \le y \le 1$$

$$P\left(\frac{1}{4} \le y \le \frac{3}{4}\right) = \int_{1/4}^{3/4} f(y)\,dy = \int_{1/4}^{3/4} \frac{6}{5}\left(\frac{1}{2} + y^2\right)dy$$

$$= \frac{6}{5}\left(\frac{1}{2}y + \frac{y^3}{3}\right)_{1/4}^{3/4}$$

$$= \frac{6}{5}\left[\frac{1}{2}\left(\frac{3}{4} - \frac{1}{4}\right) + \frac{1}{3}\left((3/4)^3 - (1/4)^3\right)\right]$$

$$= \frac{6}{5}\left[\frac{1}{2}\cdot\frac{2}{4} + \frac{1}{3}\left(\frac{26}{64}\right)\right]$$

$$= \frac{6}{5}\left(\frac{1}{4} + \frac{13}{96}\right) = \frac{6}{5}\left(\frac{24+13}{96}\right)$$

$$= \frac{6}{5} \times \frac{37}{96} = \frac{37}{80}$$

$$P\left(\frac{1}{4} \le y \le \frac{3}{4}\right) = 0.4625.$$

3. If the joint pdf of (x, y) is $f(x,y) = \begin{cases} e^{-(x+y)} & , x>0, y>0 \\ 0 & , \text{otherwise} \end{cases}$

find $P(x<1)$.

Let $f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_{0}^{\infty} e^{-(x+y)}\,dy$

$= e^{-x}\int_{0}^{\infty} e^{-y}\,dy$

$= e^{-x}\left(-e^{-y}\right)_{0}^{\infty}$

$= e^{-x}\left(-e^{-\infty} + e^{0}\right)$

$f(x) = e^{-x}, \; x>0$

Let $P(x<1) = \int_{0}^{1} f(x)\,dx$

$= \int_{0}^{1} e^{-x}\,dx$

$= \left(-e^{-x}\right)_{0}^{1}$

$= -e^{-1} + e^{0}$

$P(x<1) = 1 - e^{-1}$.

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